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The possibility of parity violation through spacetime torsion has been explored in a scenario containing fields with different spins. Taking the Kalb-Ramond field as the source of torsion, an explicitly parity violating $U(1)_{EM}$ gauge invariant theory has been constructed by extending the KR field with a Chern-Simons term.

Parity violation in gravitation is an important possibility opened up by torsion in a curved spacetime. Essentially, an antisymmetric extension of the affine connection destroys the cyclicity of the Riemann-Christoffel tensor, thereby enabling one to add a term of the form $\epsilon^{\alpha\beta\mu\nu} R_{\alpha\beta\mu\nu}$ to the Einstein-Cartan (EC) action [1]. Such a term explicitly violates parity in the gravity sector [2]. Since torsion is often looked upon as an artifact of matter fields with spin [3], it is desirable to have a consistent scheme of incorporating parity-violating effects in the coupling of particles of different spins with torsion. Such a scheme was attempted first in [4] by augmenting the EC connection with a general set of pseudo-tensorial terms constructed out of the torsion tensor itself.

Though the procedure in [4] is straightforward for the pure gravity sector and for spin-1/2 particles, it is beset with the well-known problem of gauge-invariance [5] when one takes up massless spin -1 fields. This problem was circumvented in [4] by assuming non-minimal couplings in the gauge field sector. An alternative approach was adopted in [6] where electromagnetism was incorporated in a gauge-invariant fashion into the EC framework with the help of a Chern-Simons extension. Such an extension is natural in the context of a heterotic string theory to ensure anomaly cancellation. The source of torsion in this case is the two-form Kalb-Ramond (KR) field [7] which appears in the massless sector of the heterotic string spectrum [8]. However, the possibility of parity violation was not addressed in [6].

In this note, we generalize upon the earlier works in several ways. First, the exact source of the torsion field is explored. The different couplings, especially those giving rise to parity violation, are obtained in terms of the KR fields. Secondly, the Chern-Simons term is utilized to obtain gauge-invariant interaction of the torsion field with electromagnetism. It is observed that the Chern-Simons term itself can lead to new parity-violating couplings, although they are relatively suppressed. Finally, we comment on the existence (or otherwise) of parity violation when other forms of the pseudo-tensorial extension are used. Thus we not only achieve a significant generalization of the previously obtained results, but also are able to pinpoint the characteristics which the torsion field must have in order to be a source of parity violation.

The gauge-invariant action for the EC-Maxwell-KR-fermion scenario can be written as

$$S = \int d^4x \sqrt{-g} \left[\frac{\tilde{R}(g, T)}{\kappa} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} + \frac{1}{\sqrt{\kappa}} T^{\mu\nu\lambda} \mathcal{H}_{\mu\nu\lambda} + \mathcal{L}_{fermion} \right] \quad (1)$$

where $\mathcal{T}_{\mu\nu\lambda}$ and $\mathcal{H}_{\mu\nu\lambda}$ are respectively the generalized torsion and KR field strength involving the parity-violating extensions

$$\mathcal{T}_{\nu\lambda}^{\mu} = T_{\nu\lambda}^{\mu} + q \left(\epsilon_{\nu\lambda}^{\alpha\beta} T_{\alpha\beta}^{\mu} + \epsilon_{\rho[\nu}^{\mu\sigma} T_{\lambda]\sigma}^{\rho} \right) \quad (2)$$

$$\mathcal{H}_{\nu\lambda}^{\mu} = \tilde{H}_{\nu\lambda}^{\mu} + q \left(\epsilon_{\nu\lambda}^{\alpha\beta} \tilde{H}_{\alpha\beta}^{\mu} + \epsilon_{\rho[\nu}^{\mu\sigma} \tilde{H}_{\lambda]\sigma}^{\rho} \right) \quad (3)$$

Once the covariant derivative \tilde{D}_{μ} is defined in terms of $\tilde{\Gamma}_{\mu\lambda}^{\alpha} = \Gamma_{\mu\lambda}^{\alpha} - \mathcal{T}_{\mu\lambda}^{\alpha}$ (where $\Gamma_{\mu\lambda}^{\alpha}$ is the usual Christoffel connection), the metricity condition $\tilde{D}_{\mu} g^{\mu\nu} = 0$ is automatically preserved [4].

$\tilde{H}_{\mu\nu\lambda}$, the modified KR field strength, can be expressed with a Chern-Simon extension as

$$\tilde{H}_{\mu\nu\lambda} = H_{\mu\nu\lambda} + \sqrt{\kappa} A_{[\mu} F_{\nu\lambda]} \quad (4)$$

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It remains invariant under the $U(1)$ gauge transformation $\delta A_\mu = \partial_\mu \omega$ if the KR potential transforms as $\delta B_{\mu\nu} = -\omega F_{\mu\nu}$. Here $\kappa = 16\pi G$ is the gravitational coupling constant and the electromagnetic field tensor $F_{\mu\nu}$ is assumed to be invariant under KR gauge transformation. q is a parameter determining the degree of parity-violation and it presumably depends on the matter distribution. $\tilde{R}(g, T)$ is the scalar curvature of the spacetime with torsion, defined by $\tilde{R} = \tilde{R}_{\alpha\mu\beta\nu} g^{\alpha\beta} g^{\mu\nu}$. $\tilde{R}_{\alpha\mu\beta\nu}$ is the Riemann-Christoffel tensor:

$$\tilde{R}_{\mu\nu\lambda}^\rho = \partial_\mu \tilde{\Gamma}_{\nu\lambda}^\rho - \partial_\nu \tilde{\Gamma}_{\mu\lambda}^\rho + \tilde{\Gamma}_{\mu\sigma}^\rho \tilde{\Gamma}_{\nu\lambda}^\sigma - \tilde{\Gamma}_{\nu\sigma}^\rho \tilde{\Gamma}_{\mu\lambda}^\sigma \quad (5)$$

$\tilde{R}(g, T)$ is related to the Einstein scalar curvature R as

$$\tilde{R}(g, T) = R(g) - T_{\mu\nu\lambda} T^{\mu\nu\lambda} \quad (6)$$

$\mathcal{L}_{fermion}$ being the Lagrangian density for a Dirac fermion.

The role of the augmented KR field strength three-tensor as the spin angular momentum density (which is the source of torsion [9]) is evident from Eq.(1) where the torsion tensor $T_{\mu\nu\lambda}$, being an auxiliary field, obeys the constraint equation

$$T_{\mu\nu\lambda} = \sqrt{\kappa} \mathcal{H}_{\mu\nu\lambda} \quad (7)$$

and the same relation holds between T and \tilde{H} as well.

The action for the gravity sector is

$$S_G = \int d^4x \sqrt{-g} \tilde{R} \quad (8)$$

where \tilde{R} includes the contributions due to torsion, and is given by

$$\tilde{R}(g, \mathcal{H}) = R(g) - \kappa \tilde{H}_{\mu\nu\lambda} \tilde{H}^{\mu\nu\lambda} - 6\kappa q \epsilon_{\alpha\beta}^{\nu\lambda} \tilde{H}_{\nu\lambda}^\mu \tilde{H}_\mu^{\alpha\beta} \quad (9)$$

The second term corresponds to the Einstein-Cartan extension of the general theory of relativity, which conserves parity. The last term which is parity-violating is identical with that given in [4]. In terms of the KR field strength and the Maxwell field, the parity-violating term in the above equation can be expressed as

$$R^{pv} = 6\kappa q [\epsilon_{\alpha\beta}^{\nu\lambda} H_{\mu\nu\lambda} (H^{\mu\alpha\beta} + 2\sqrt{\kappa} [A^\mu F^{\alpha\beta} + 2 A^\alpha F^{\beta\mu}]) + \kappa (A_\mu F_{\nu\lambda} + 4 A_\lambda F_{\mu\nu}) A^\mu * F^{\nu\lambda}] \quad (10)$$

where $*F^{\nu\lambda} = \epsilon^{\nu\lambda\alpha\beta} F_{\alpha\beta}$ is the (*Hodge*-) dual of $F_{\alpha\beta}$. The field equations that can be obtained on extremizing the action (8) with respect to $g_{\mu\nu}$ are the usual Einstein's equations in the present picture:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa \tau_{\mu\nu} \quad (11)$$

where the symmetric two-tensor $\tau_{\mu\nu}$ has direct analogy with the energy-momentum tensor for matter couplings and is a clear manifestation of spin-density in an extended Einstein-Cartan spacetime. It has the general form [10]

$$\tau_{\mu\nu} = \left(3g_{\nu\rho} \mathcal{H}_{\alpha\beta\mu} \mathcal{H}^{\alpha\beta\rho} - \frac{1}{2} g_{\mu\nu} \mathcal{H}_{\alpha\beta\gamma} \mathcal{H}^{\alpha\beta\gamma} \right) \quad (12)$$

The EC-KR theory can be made to couple with electromagnetism in a gauge invariant manner in the following way [6]:

$$\mathcal{L}_{G-EM} = \frac{R}{\kappa} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \mathcal{H}_{\mu\nu\lambda} \mathcal{H}^{\mu\nu\lambda} \quad (13)$$

whence the modified Maxwell equations can be obtained as

$$D_\mu F^{\mu\nu} = \sqrt{\kappa} \mathcal{H}^{\mu\nu\lambda} F_{\lambda\nu} \quad (14)$$

In addition to the above equations, the Maxwell-Bianchi identity

$$D_\mu *F^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} *F^{\mu\nu}) = 0 \quad (15)$$

also holds for the electromagnetic field. The KR-Maxwell interaction term in the action is given by

$$S_{int} = \int d^4x \sqrt{-g} [H_{\mu\nu\lambda} (3 A^\mu F^{\nu\lambda} + 6q A^\mu *F^{\nu\lambda} + 12q \epsilon_{\alpha\beta}^{\nu\lambda} A^\alpha F^{\beta\mu})] \quad (16)$$

The second and third terms in the expression are the exclusive parity-violating modifications over the result found in [6]. If one further expresses the KR field strength as the (*Hodge*-) dual of the derivative of a pseudoscalar field, the only non-vanishing parity-violating contributions can come from the Chern-Simons extension which, however, are suppressed by the square of the Planck mass.

For the spin 1/2 fermion, the extended Dirac Lagrangian has the form [11,12]

$$\mathcal{L}_{fermion} = \bar{\psi} \left[i\gamma^\mu \left(\partial_\mu - \sigma^{\rho\beta} v_\rho^\nu g_{\lambda\nu} \partial_\mu v_\beta^\lambda - g_{\alpha\delta} \sigma^{ab} v_a^\beta v_b^\delta \tilde{\Gamma}_{\mu\beta}^\alpha \right) \right] \psi \quad (17)$$

where the tetrad v_b^λ connects the curved space (designated by Greek indices) to the corresponding tangent space (designated by Latin indices) at any point. As shown in [4], using the full form of $\tilde{\Gamma}$, $\mathcal{L}_{fermion}$ can be written

$$\mathcal{L}_{fermion} = \mathcal{L}_f^E + \mathcal{L}_f^C + \mathcal{L}_f^{pv} \quad (18)$$

where \mathcal{L}_f^E is the Dirac Lagrangian in Einstein gravity, \mathcal{L}_f^C is its Cartan extension which is parity-conserving and can be expressed in terms of the KR field and the Maxwell field (following the identification (7)) as

$$\mathcal{L}_f^C = \bar{\psi} \left[i\gamma^\mu \sigma^{ab} v_a^\beta v_b^\omega (\sqrt{\kappa} H_{\beta\omega\mu} + \kappa (A_\mu F_{\beta\omega} + 2 A_\beta F_{\omega\mu})) \right] \psi \quad (19)$$

The additional term which is responsible for parity violation and is denoted here by \mathcal{L}_f^{pv} , has the form

$$\begin{aligned} \mathcal{L}_f^{pv} = & q\sqrt{\kappa} \bar{\psi} \left[i\gamma^\mu \sigma^{ab} v_a^\beta v_b^\omega (\epsilon_{\mu\beta}^{\gamma\delta} H_{\omega\gamma\delta} + \epsilon_{\omega[\mu}^{\gamma\delta} H_{\beta]\gamma\delta}) \right] \psi \\ & + q\kappa \bar{\psi} \left[i\gamma^\mu y \sigma^{ab} v_a^\beta v_b^\omega \left(2 A_\omega *F_{\mu\beta} - A_\mu *F_{\omega\beta} + 4 \epsilon_{\mu\beta}^{\gamma\delta} A_\gamma F_{\delta\omega} - 2 \epsilon_{\omega\beta}^{\gamma\delta} A_\gamma F_{\delta\mu} \right) \right] \psi \end{aligned} \quad (20)$$

Finally, it is worth mentioning here that the metricity condition imposes no compelling need to take the generalized torsion tensor $\mathcal{T}_{\nu\lambda}^\mu$ to be antisymmetric in all its three indices; an antisymmetry in only a pair of indices will suffice. As such, $\mathcal{T}_{\nu\lambda}^\mu$ can be expressed in a more general way:

$$\mathcal{T}_{\nu\lambda}^\mu = \mathcal{T}_{\nu\lambda}^\mu + q_1 \epsilon_{\nu\lambda}^{\alpha\beta} \mathcal{T}_{\alpha\beta}^\mu + q_2 \epsilon_{\rho[\nu}^{\mu\sigma} \mathcal{T}_{\lambda]\sigma}^\rho \quad (21)$$

where the constants q_1 and q_2 are in general different, and antisymmetry is confined to the two lower indices (although $\mathcal{T}^{\mu\nu\lambda}$ is still antisymmetric in all three indices). In the gravity sector, we then have a Lagrangian density given by

$$\mathcal{L}_{gravity} = \tilde{R}(g, T) = R(g) - \partial^\lambda \mathcal{T}_{\alpha\lambda}^\alpha + g^{\mu\lambda} \Gamma_{\mu\lambda}^\rho \mathcal{T}_{\alpha\rho}^\alpha + \mathcal{T}_\rho^{\alpha\lambda} \mathcal{T}_{\alpha\lambda}^\rho \quad (22)$$

which, in terms of T , can be expressed as

$$\mathcal{L}_{gravity} = \tilde{R}(g, T) = R(g) + R^{pc}(T) + R^{pv}(\epsilon, T) \quad (23)$$

with

$$R^{pc}(T) = - [1 + 2(q_1 + 2q_2)] T_{\nu\lambda}^\mu T_\mu^{\nu\lambda} \quad (24)$$

$$R^{pv}(\epsilon, T) = - 2(q_1 + 2q_2) \epsilon_{\beta\sigma}^{\rho\alpha} T_{\rho\alpha}^\lambda T_\lambda^{\beta\sigma} - (q_1 - q_2) \left[\partial^\lambda (\epsilon_{\sigma\lambda}^{\alpha\beta} T_{\alpha\beta}^\sigma) - g^{\mu\lambda} \Gamma_{\mu\lambda}^\rho \epsilon_{\sigma\rho}^{\alpha\beta} T_{\alpha\beta}^\sigma \right] \quad (25)$$

The presence of the derivatives of T makes it dynamic, i.e., the torsion tensor $T_{\mu\nu\lambda}$ is no longer an auxiliary field for $q_1 \neq q_2$. Hence, the identification of T with H is not possible anymore. The spin - 1/2 fermion field in this case has the Lagrangian density

$$\mathcal{L}_{fermion} = \mathcal{L}_f^E(g) + \mathcal{L}_f^{pc}(T) + \mathcal{L}_f^{pv}(\epsilon, T) \quad (26)$$

where $\mathcal{L}_f^E(g)$ includes the set of terms corresponding to Einstein gravity, and

$$\mathcal{L}_f^{pc}(T) = \bar{\psi} \left[i\gamma^\mu \sigma^{ab} v_a^\beta v_b^\omega T_{\mu\beta\omega} \right] \psi \quad (27)$$

$$\mathcal{L}_f^{pv}(\epsilon, T) = q_1 \bar{\psi} \left(i\gamma^\mu \sigma^{ab} v_a^\beta v_b^\omega \epsilon_{\mu\beta}^{\alpha\delta} T_{\alpha\delta\mu} \right) \psi + q_2 \bar{\psi} \left(i\gamma^\mu \sigma^{ab} v_a^\beta v_b^\omega \epsilon_{\omega[\mu}^{\lambda\nu} T_{\beta]\lambda\nu} \right) \psi \quad (28)$$

which explicitly violates parity.

One interesting consequence of the above is worth noting here. As has been already pointed out, there is an alternative way of expressing the tensor field H in terms of a pseudoscalar field ϕ through a duality transformation:

$$H^{\mu\nu\lambda} = \epsilon^{\mu\nu\lambda\alpha} \partial_\alpha \phi \quad (29)$$

In such a case, the pseudo-tensorial extension of torsion vanishes as soon as one sets $q_1 = q_2$; in other words, there is no parity violation under this condition unless one uses a Chern-Simons extension. On the contrary, the inequality of the two charges retains parity-violating terms in all sectors of the Lagrangian even when torsion is expressed in terms of the dual field.

We conclude by observing that predictions have already been made linking a space-time with torsion with phenomena such as the rotation of the plane of polarization in radio waves coming from distant galactic sources [13]. Moreover, couplings arising from torsion can affect the helicity flip of neutrinos [14,15], which has far-reaching implications in the context of the solar and atmospheric neutrino anomalies. Further insight on parity violation due to torsion will help us in evolving better understanding of these phenomena.

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